Sedimentation of Dilute Suspensions in Creeping Motion

JACK FAMULARO and JOHN HAPPEL
New York University, New York, New York

Solutions to the creeping motion equations for a particle in a cylinder and particles in an infinite media were superimposed to calculate the first-order correction to the Stokes settling velocity for a particle inside a cylindrical container and for various models used to approximate the settling characteristics of dilute suspensions. The models were based on an assumed distribution of translating particles subject only to the hydrodynamic forces produced by their motion. Particle interactions and wall effects for ordered and random arrangements of spherical particles in a cylinder were calculated on a digital computer and used to develop equations for settling velocity as a function of concentration.

The equations developed for the models containing spherical particles were modified to apply to nonspherical particles by considering each particle as a point force with regard to the fluid motion it produces at neighboring particles. As an application of these equations, a model of a dilute fiber suspension was studied in which prolate spheroids settle perpendicular to their symmetry axes.

The correction to Stokes' law for ordered arrangements of spheres is close to that obtained by earlier investigators working with similar models. However this correction leads to lower settling rates than the bulk of reported experimental data at any given particle concentration in the dilute range of less than 5% solids. The authors have shown that this discrepancy may be eliminated by assuming an alternate model composed of doublets in place of uniformly distributed particles. It may be concluded that in the dilute concentration range a model of a sedimenting suspension must include terms which reflect its tendency toward cluster formation and the consequent effect on settling rate. In order to accomplish this, forces other than hydrodynamic may have to be considered, for example, electrostatic forces.

Since the publication of Stokes' work (*17*) on low Reynolds number flow past a spherical particle, a large body of literature has evolved on the derivation of corrections to Stokes' law when the particle is in the vicinity of a boundary. This is precisely the condition which exists in a suspension where each particle is affected by neighboring particles and by container walls. It is not surprising to note that large deviations from Stokes' law are possible due to these interactions, even in creeping motion.

The method described in this paper for obtaining the correction to Stokes' settling velocity for ordered suspensions is similar to that employed by Burgers (*3*). Particle interactions were obtained by superimposing velocity fields to cancel successively the motion at particle sites (method of reflections). Burgers introduced the wall effect by superimposing an upward return flow through the suspension to cancel the downward volumetric flow rate. In the present work the wall effect was evaluated in exactly the same manner as particle interaction; that is, by successively cancelling the motion at the particles and container wall. In this sense the equations derived here do not depend on assumptions regarding the nature of the flow field, as is true in Burgers' work.

Burgers also studied random suspensions, using an assumed particle distribution which assigned zero probability to finding a particle closer than one diameter from an object particle and equal probability outside this region. He obtained a correction to the Stokes' settling velocity proportional to the first power of particle concentration. This differs from his treatment of ordered suspensions in which a correction in terms of the one-third power of particle concentration is reported. The difference follows directly from the assumed probability distribution. Random suspensions discussed in this paper were constructed with a uniform random number subroutine on a digital computer and then treated in the same manner as ordered suspensions. The resultant correction to Stokes' law is proportional to the one-third power of particle concentration, and the proportionality factor is not very different from that obtained for cubic and rhombohedral suspensions, also studied.

Theoretical treatments of sedimentation in dilute systems do not agree with most experimental data (*4, 11, 14*). Observed settling rates tend to be greater than theoretical predictions. This is also true of the present work on uniform suspensions. In dilute suspensions, where the free volume is large compared to that occupied by particles, there may be considerable tendency for particles to agglomerate, which always results in reduced resistance due to by-passing fluid motion. This phenomenon is illustrated in a discussion of settling rates for suspensions of doublets and emphasizes the necessity of developing settling rate equations in terms of extent of agglomeration for dilute systems.

MATHEMATICAL DEVELOPMENT

The mathematical problem posed by the slow settling of a suspension of spherical particles involves the solution of the creeping motion equations:

$$\mu \nabla^2 \mathbf{v} = \text{grad } p \qquad (1)$$

$$\text{div } \mathbf{v} = 0 \qquad (2)$$

subject to boundary conditions of zero slip velocity on all solid surfaces. The above equations are valid only for low particle Reynolds number.

The linearity of the creeping motion equations permits the construction of a solution to the boundary value problem by superposition of velocity fields which are separate solutions of the creeping motion equations. Each individual velocity field is equal to the negative of the previous sum of velocity fields acting at a solid surface, thereby temporarily satisfying the boundary condition of zero slip at that surface.

Since we are interested in computing a first-order correction to Stokes' law, we will require only the first velocity field to be superimposed at each solid surface.

Figure 1 contains a schematic diagram of the velocity fields of interest in obtaining a first-order correction to the Stokes' drag on a particle i in a suspension of n particles.

In Figure 1 the velocity field denoted by ω_{ji} is the axial component of the motion resulting from the settling of sphere j in an infinite media, assuming the absence of all other particles. The second subscript i indicates that this field is evaluated at the center of sphere i. The field W_{ji} has the property that it is equal to $-\omega_{ji}$ on the surface of the cylindrical container. It should be noted that in terminating the reflection procedure after addition of the fields W_{ki}, the boundary condition of zero slip velocity is satisfied on the surface of the container. However, a residual motion

$$\sum_{\substack{k=1\\k\neq i}}^{n} \omega_{ki} + \sum_{k=1}^{n} W_{ki}$$

exists at the center of particle i. The magnitude of this residual motion can be made arbitrarily small by reducing the ratio of particle size to interparticle spacing. Thus the first reflection solution becomes increasingly more accurate as particle concentration is reduced.

The velocity field ω_{ji} is the axial component of the well-known Stokes' velocity field (*11*) and is given by

$$\omega_{ji} = 3/4 \frac{a}{L_{ji}} U \left[1 + \left(\frac{z_{ji}}{L_{ji}} \right)^2 \right] + O\left[\left(\frac{a}{L_{ji}} \right)^3 \right] \quad (3)$$

where a is the particle radius, L_{ji} is the distance between particles, U is the settling speed, and z_{ji} is the axial component of L_{ji}. For dilute systems the term of order $(a/L_{ji})^3$ is omitted, and ω_{ji} is the same as the field induced by a point force.

The velocity field W_{ji}, evaluated at the center of particle j, is the axial component of the first reflected velocity field from sphere j settling in an eccentric position inside a cylindrical container. This field has been evaluated by Brenner and Happel (*1*) and is reported [Equation (4)] in a modified form suitable for computation. Consult

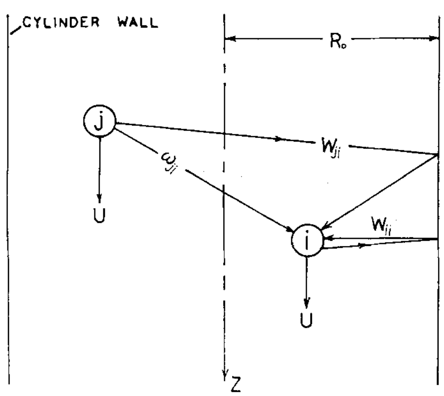

Fig. 1. Schematic diagram of velocity fields required to obtain a first-order correction to Stokes' resistance.

reference 6 for the mathematical analysis required to obtain Equation (4) and for the exact form of the functions $f(\lambda)$, $g(\lambda)$, and $h(n, \lambda)$.

$$W_{ji} = -\frac{3}{2\pi} \frac{a}{R_o} U \left[\int_0^\infty \frac{f(\lambda)}{Z} \sin(\lambda Z)\, d\lambda + \int_0^\infty g(\lambda) \cos(\lambda Z)\, d\lambda + 2 \sum_{n=1}^{\infty} \cos(n\phi) \int_0^\infty h(n, \lambda) \cos(\lambda Z)\, d\lambda \right] \quad (4)$$

The resistance experienced by sphere i in a suspension of n particles settling at the rate U now can be obtained by substituting the net fluid motion at the center of particle i in place of the uniform field U in Stokes' equation for the drag on a spherical particle. Thus

$$D_i = 6\pi\mu a \left(U - \sum_{\substack{j=1\\j\neq i}}^{n} \omega_{ji} - \sum_{j=1}^{n} W_{ji} \right) \quad (5)$$

This equation ignores the gradient in the fluid velocity field at the center of particle i. For a first reflection approach restricted to dilute suspensions, this gradient term will be small compared to the velocity at the center of the particle.

The fields ω_{ji} are in the direction of the positive axis, are positive, and hence reduce the resistance. The fields W_{ji} represent the backflow through the suspension and hence increase the resistance.

The distances z_{ji} and L_{ji} in Equation (3) may be divided by the cylinder radius R_o to give

$$\omega_{ji} = \frac{a}{R_o} U \frac{3}{4(L_{ji}/R_o)} \left[1 + \left(\frac{z_{ji}/R_o}{L_{ji}/R_o} \right)^2 \right]$$

Upon comparison with Equation (4), we see that the term $\frac{a}{R_o} U$ may be factored from ω_{ji} and W_{ji}. Accordingly, we may define a coefficient σ by the equation

$$\sigma = \frac{-\sum_{j=1\neq i}^{n} \omega_{ji} - \sum_{j=1}^{n} W_{ji}}{\frac{a}{R_o} U} \quad (6)$$

σ will be called a *particle size coefficient* to emphasize that the selection of a particle radius in conjunction with σ is sufficient to compute the correction to Stokes' law.

Equation (5) may now be expressed in terms σ as

$$D_i = 6\pi\mu a U \left[1 + \sigma \frac{a}{R_o} \right] \quad (7)$$

It is customary to represent the correction to Stokes' law in terms of the particle concentration in the suspension. The particle concentration will be denoted by C and defined as the ratio of the volume of particles present in the suspension divided by the total volume occupied by particles and fluid. This is equal to one minus the void volume. For a given total volume the particle concentration is dependent on the volume of solids present. The volume of solids is, in turn, a function of the number of particles and the size of each particle. The number of particles per unit volume of suspension will be designated α. In the remainder of this paper α will be taken as a measure of the *density* of a suspension, where density implies number of particle sites, without regard to size. In accordance with the above discussion we have

$$C = \alpha\, 4/3\, \pi\, a^3 = (\alpha\, R_o^3)\, 4/3\, \pi \left(\frac{a}{R_o}\right)^3$$

which may also be written

$$\frac{a}{R_o} = \left[\frac{3}{4\,\pi\,(\alpha\, R_o^3)}\right]^{1/3} C^{1/3} \qquad (8)$$

The product $(\alpha\, R_o^3)$ represents the number of particles present in a volume equal to the cylinder radius cubed.

We may now define a concentration coefficient by

$$\gamma = \left[\frac{3}{4\,\pi\,(\alpha\, R_o^3)}\right]^{1/3} \sigma \qquad (9)$$

and substitute into Equation (7) to give

$$D_i = 6\,\pi\,\mu\, a\, U\,(1 + \gamma\, C^{1/3}) \qquad (10)$$

COMPUTATION OF SETTLING VELOCITY

Numerical Evaluation of Integrals in W_{ji}

The integrals in Equation (4) offer some difficulty for the ordinary quadrature formula, because for large z, the integrands oscillate rapidly. A technique described by Filon (*18*), which compensates for the oscillatory character of the intergrand, was employed in this work. Filon's quadrature formula is a correction to Simpson's rule and is shown below:

$$\int_a^b f(\lambda)\cos(\lambda\, z)\, d\lambda \approx \Delta\lambda\,\{\alpha\,[f(b)\sin(bz) - f(a)\sin(az) + \beta\, C_{2s} + \gamma\, C_{2s-1}]\} \qquad (11)$$

where

C_{2s} = sums of even ordinates of $f(\lambda)\cos(\lambda z)$ less half the first and last ordinates,

C_{2s} = sum of odd ordinates,

α, β, γ = functions of $z\,\Delta\,\lambda$

Similarly

$$\int_a^b f(\lambda)\sin(\lambda\, z)\, d\lambda \approx \Delta\lambda\,\{-\alpha\,[f(b)\cos(bz) - f(a)\cos(az) + \beta\, S_{2s} + \gamma\, S_{s2-1}]\} \qquad (12)$$

Details of the digital computer program in the Fortran language (*13*) for the computation of the integrals in Equation (4) are presented in reference *6*. Errors due to truncation of the integration before an upper limit, $b = \infty$, and due to a finite interval size $\Delta\lambda$, were evaluated by trial on the computer by increasing successively b and reducing $\Delta\lambda$. Error bounds indicated in this paper are conservative estimates based on this procedure.

Effect of Eccentricity on the Settling Velocity of a Single Particle

Equation (7) can be adapted to the computation of the resistance experienced by a single eccentric particle, in which case

$$\sigma \frac{a}{R_o} U = -W_{ii} \qquad (13)$$

The ratio of the radial coordinate of the particle divided by the cylinder radius will be called the radial eccentricity and will be designated β. In the case of a single particle the particle size coefficient σ is a function of β alone. This point is stressed by letting $\sigma = f(\beta)$. Thus Equation (7) is written

$$D_i = 6\,\pi\,\mu\, a\, U\left[1 + f(\beta)\frac{a}{R_o}\right] \qquad (14)$$

Equation (14) remains a valid first-order correction to the Stokes' drag as $\beta \to 1$, provided the ratio $a/R_o(1-\beta)$ remains small. The quantity $R_o(1-\beta)$ is the distance from the sphere center to the cylinder wall.

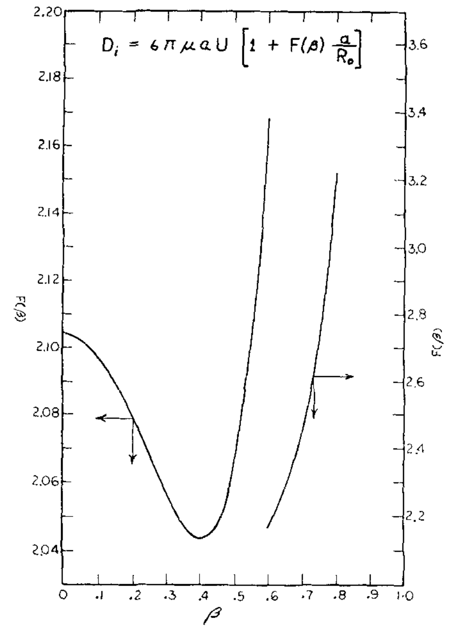

Fig. 2. Effect of eccentricity on resistance to a single sphere [Equation (14)].

If the observer of the particle settling in the cylinder is of the same size as the particle radius a, then the process of letting $\beta \to 1$, while keeping $a/R_o(1-\beta) <<< 1$, will tend to reduce the curvature of the cylinder wall. The problem then becomes equivalent to the settling of a small sphere parallel to an infinite flat plate at a large distance. Such a problem was solved by Lorentz (*12*) with the result

$$D_i = 6\,\pi\,\mu\, a\, U\left(1 + 9/16\frac{a}{R_o(1-\beta)}\right) \qquad (15)$$

Comparison with Equation (14) reveals that a check on the validity of $f(\beta)$ is obtained if

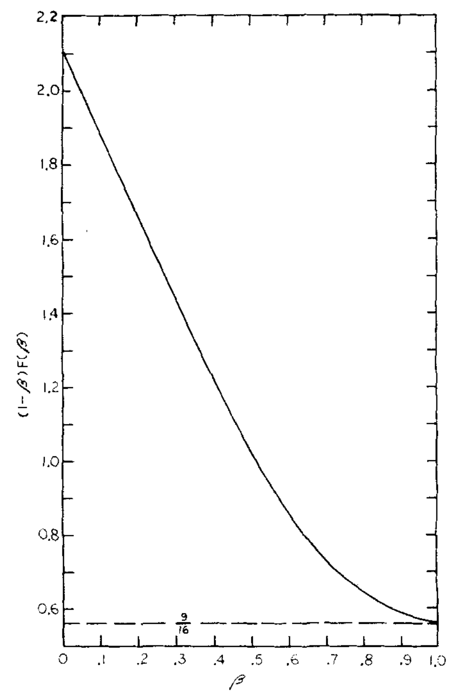

Fig. 3. Approach to Lorentz coefficient as $\beta \to 1$ for a single sphere [Equation (14)].

TABLE 1. CONCENTRATION COEFFICIENTS FOR CUBIC ARRAYS

Suspension density αR_o^3	Concentration coefficient (interface) γi	Concentration coefficient (center) γ
172	1.89	1.93
288	1.87	1.91
423	1.88	1.93
672	1.86	1.90
Average	1.88	1.92

$$\lim_{\beta \to 1} (1 - \beta) f(\beta) = 9/16 \tag{16}$$

Figures 2 and 3 contain the results of computations at various values of β. From Figure 3 it is apparent that the limit expressed by Equation (16) is in the close vicinity of the extrapolated curve of $(1 - \beta)f(\beta)$. In addition we observe a minimum resistance to settling at $\beta \approx 0.4$ in Figure 2.

It is a simple matter to deduce that the ratio of particle settling velocity U to Stokes' settling velocity U_o is given by

$$\frac{U}{U_o} = \frac{1}{1 + f(\beta)\dfrac{a}{R_o}} \tag{17}$$

and that the maximum value is

$$\left(\frac{U}{U_o}\right)_{max} \approx \frac{1}{1 + 2.044\dfrac{a}{R_o}} \tag{18}$$

occurring at $\beta \approx 0.4$.

Settling Velocity of Cubic Suspensions

A number of models were postulated to approximate the settling characteristics of real suspensions. The location of particles in the suspension was preset and the settling rate was computed by summing particle interactions and wall effects. Hydrodynamic forces alone were considered.

The first model studied consisted of spherical particles positioned at the corners of cubic cells, the entire lattice confined inside a cylindrical container. The concentration coefficient γ was computed for particles at the longitudinal center of an infinite suspension and at the interface of a semi-infinite suspension. The results of this computation conducted at four different lattice refinements are shown in Table 1.

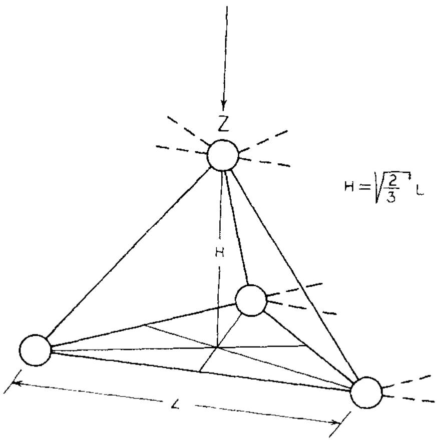

Fig. 4. Rhombohedral array.

If the settling rate equation for the cubic model is based on centrally located particles, it follows from Equation (10) that

$$\frac{U}{U_o} = \frac{1}{1 + 1.92\, C^{1/3}} \tag{19}$$

The last place in the coefficient γ is in doubt by two units; that is $\gamma = 1.92 \pm 0.02$.

Many experimental studies of sedimentation are performed by observing the settling rate of the top interface of the suspension. The close correspondence between values of γ at the interface and center of the cubic array would suggest that this is a permissible procedure.

Settling Velocity of Rhombohedral Suspensions

In order to establish whether particle arrangement can influence the value of γ for uniform, ordered suspensions, a program was written to position particles in a rhombohedral array as shown in Figure 4.

In such an array particles are positioned at the vertices of a tetrahedron whose faces are equilateral triangles. Horizontal layers contain particles at the corners of equilateral triangles of side L and adjacent layers are a vertical distance H apart.

A single rhombohedral suspension was treated containing 1,523 spheres per cylinder radius cubed. The computed value of γ for this suspension was $\gamma = 1.79 \pm 0.02$; hence the settling rate equation for rhombohedral suspensions is

$$\frac{U}{U_o} = \frac{1}{1 + 1.79\, C^{1/3}} \tag{20}$$

The small difference between the value of γ for cubic and rhombohedral suspensions suggests that particle arrangement is not very important in uniform suspensions. To explore this premise further we modified the rhombohedral program to space adjacent horizontal layers at a variable distance apart. At $H = L$, the ratio of vertical to horizontal spacing between particles is unity as in cubic suspensions. For this H a value of $\gamma = 1.88$ was computed which is quite close to $\gamma = 1.92$ for cubic suspensions. In the reverse direction adjacent horizontal layers were placed closer together than in the rhombohedral array. The results of this study are plotted in Figure 5 where it is clearly evident that vertical compression of the suspension has very little influence on γ in the range $0.2 \leq H/L \leq 1$. A sharp dip in γ is noted for $H/L < 0.2$.

Settling Velocity of Random Suspensions

The arrangement of particles in a real suspension never can achieve the orderliness of cubic or rhombohedral arrays. Therefore it is of interest to inquire into the set-

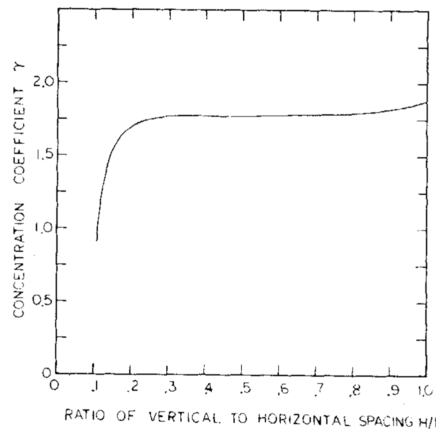

Fig. 5. Variation of γ with H/L for deformed rhombohedral suspensions.

tling characteristics of a model which has the quality of disarrangement observed in real suspensions. To this end suspensions were synthesized in which particles were located at random about a central object particle. The positioning program was such that particles were regarded as points and therefore could assume any position with equal probability. The justification for such a treatment lies in the fact that a first-order correction to Stokes' law is actually a point force approach. Thus particle size can always be considered small in relation to the smallest interparticle spacing.

In order to compare random suspensions with ordered suspensions in which the dependence of settling rate on concentration is fixed, it was decided to synthesize random suspensions containing the same number of particles as the cubic suspension with suspension density, $(\alpha R_o^3) = 672$ spheres per R_o^3. This was the densest cubic suspension studied. In principle every particle in the suspension, except for a centrally located particle, could have been positioned at random and the settling rate of the central particle could have been computed with the same programs used for the cubic suspension. However, such an approach would have been exceedingly wasteful of computer time. Instead, a series of calculations, described in detail in reference *6*, was performed to demonstrate that the major influence of orientation on γ was due to the random positioning of particles in a small volume surrounding the center particle. This volume was equivalent roughly to the segment of the cubic suspension $(\alpha R_o^3 = 672)$ consisting of sixty-three cubic cells in the form of a $3 \times 3 \times 7$ cell parallelepiped. Outside of this volume the mean contribution to γ from randomly positioned particles was shown to be very close to the contribution from particles in the cubic array. Thus it was concluded that the relationship between settling rate and concentration for random suspensions could be obtained by positioning sixty-two particles at random about a center object particle in the $3 \times 3 \times 7$ parallelepiped, and by employing the cubic suspension contribution to γ for particles outside this parallelepiped. The sixty-two random locations were obtained with a uniform random number subroutine on a digital computer.

Because of the random arrangement of the sixty-two particles under observation, a wide variation in γ resulted. In a total of two hundred thirty-one different random configurations the mean value of γ was 1.30. It is shown (*5*) that this mean value has 99.5% confidence limits of ± 0.24; that is

$$\gamma = 1.30 \pm 0.24 \tag{21}$$

Thus for random suspensions

$$\frac{U}{U_o} = \frac{1}{1 + 1.3\ C^{1/3}} \tag{22}$$

A lower value of the concentration coefficient γ for random suspensions is quite plausible, because of the nature of the velocity fields contributing to γ in Equation (12). In a sample of random configurations a certain number will contain particles closer to the center particle than the mean interparticle spacing. This will tend to increase the settling rate of the center particle and hence reduce γ. In the reverse direction a certain number of configurations will be such that the closest particle to the center particle will be at a greater distance than the mean spacing. This will increase γ. However, the rate of change of γ is greatest in the vicinity of the object particle. Hence, for a prescribed displacement toward the object particle, the reduction in γ will be greater than the increase in γ due to an equal displacement away from the object particle. The net effect of the random arrangement of sixty-two particles is therefore a reduction in γ.

Extension to Suspensions of Arbitrary Particles

Although the preceding discussion pertains to suspensions of spherical particles, under certain circumstances the results may also be applied to particles of arbitrary shape. Brenner (*2*) has shown that when a single arbitrary particle settles inside a container in such an orientation that the drag force is parallel to the motion

$$\frac{D'}{D_o'} = \frac{1}{1 - \dfrac{W'}{U'}} \tag{23}$$

Applied to situations where the particle size is small compared to its distance from the container walls, W'/U' is small and Equation (23) may be replaced by the approximate form.

$$D' = D_o' \left[1 + \frac{W'}{U'} \right] \tag{24}$$

In order to adapt Equation (24) to a dilute suspension of arbitrary particles, the field W' must be replaced by the total motion at the center of a particle i in a suspension of n particles
Thus

$$D_i' = D_o' \left[1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\omega'_{ji}}{U'} - \sum_{j=1}^{n} \frac{W'_{ji}}{U'} \right] \tag{25}$$

The fields ω'_{ji} and W'_{ji} are those shown in Figure 1, if the spheres are replaced by arbitrary particles. Equation (25) is the analog of Equation (8) for arbitrary particles.

If we restrict our results to dilute suspensions, the point force solution of Lamb (*12*) may be employed for ω'_{ji}.

$$\omega'_{ji} = \frac{D_o'}{8\,\pi\,\mu\,L_{ji}} \left[1 + \left(\frac{z_{ji}}{L_{ji}} \right)^2 \right] \tag{26}$$

Equation (26) may now be compared with Equation (3), noting that

$$D_o = 6\,\pi\,\mu\,a\,U_o$$

for a sphere. It then follows that

$$\omega'_{ji} = \omega_{ji} (D_o'/D_o)$$

and

$$W'_{ji} = W_{ji} (D_o'/D_o)$$

Hence Equation (25) may be written

$$D_i' = D_o' \left[1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\omega_{ji} D_o'}{U' D_o} - \sum_{j=1}^{n} \frac{W_{ji} D_o'}{U' D_o} \right] \tag{27}$$

After we factor $D_o'/U'D_o$ the residual motion at the center of particle i is obtained from Equation (9) to give

$$D_i' = D_o' \left[1 + \frac{\sigma}{6\pi\mu} \frac{D_o'}{U'R_o} \right] \tag{28}$$

In Equation (28) the coefficient σ is that computed for a suspension of spherical particles. The drag D_o' will contain a characteristic dimension of the arbitrary particle which may be employed in many cases to relate $D_o'/U'R_o$ to particle concentration.

In order to illustrate the application of Equation (28), we shall derive settling rate equations for suspensions of vertically oriented doublets and ellipsoids of revolution settling perpendicular to their axes of symmetry. The latter case may be employed as a model for fibrous systems

Table 2. Resistance Coefficient λ for a Vertical Doublet

Sphere diameter / Center-center distance, d/L_D	Resistance coefficient, λ
0	1.0
0.266	0.836
0.648	0.702
1.0	0.645

when the semimajor axes of the ellipsoids are large compared to their semiminor axes, and the distance between fibers is large compared to their length.

First, with regard to a suspension of doublets, we shall assume that the distance between spherical particles comprising a doublet is fixed and small compared to the distance between adjacent doublets. The drag experienced by a doublet settling in an unbounded medium has been obtained by Stimson and Jeffrey (*16*). Their equation for drag may be written

$$D_o' = 12\,\pi\,\mu\,a\,\lambda\,U_o' \qquad (29)$$

where λ is an analytic function of the ratio of sphere diameter to center-to-center distance within a doublet. Equation (29) represents the drag experienced by the entire doublet, considered to be a single arbitrary particle. The drag experienced by one of the spheres comprising the doublet is half of D_o' given above.

Typical values of λ are given in Table 2.

If we now consider the basic doublet to be one of many such particles settling at the rate U' in a suspension, the uncorrected drag D_o' acting on each doublet is

$$D_o' = 12\,\pi\,\mu\,a\,\lambda\,U'$$

This is substituted into Equation (28) to give

$$D_i' = 12\,\pi\,\mu\,a\,\lambda\,U'\left[\,1 + 2\lambda\sigma\frac{a}{R_o}\right] \qquad (30)$$

If Equation (9) is applied to doublets it can be shown that

$$\sigma\frac{a}{R_o} = 2^{-1/3}\,\gamma\,C^{1/3} \qquad (31)$$

Hence

$$D_i' = 12\,\pi\,\mu\,a\,\lambda\,U'\,[1 + 2.84\,\lambda\,\gamma\,C^{1/3}] \qquad (32)$$

It follows from (32) that the ratio of settling velocity of a suspension of doublets to that of a single sphere in an infinite medium is given by

$$\frac{U'}{U_o} = \frac{1}{\lambda(1 + 2.84\,\lambda\,\gamma\,C^{1/3})} \qquad (33)$$

With regard to a suspension of ellipsoids of revolution with semimajor axis ζ semiminor axis η, and $\phi = \zeta/\eta$, Happel and Brenner (*8*) gave the following equation for D_o':

$$D_o' = 6\,\pi\,\mu\,\eta\,\tau(\phi)\,U' \qquad (34)$$

where

$$\tau(\phi) = \frac{8}{3}\left[\frac{\phi}{\phi^2-1} + \frac{2\phi^2-3}{(\phi^2-1)^{3/2}}\ln(\phi + \sqrt{\phi^2-1})\right]^{-1} \qquad (35)$$

for $\phi > 1$ and the product $\eta\tau(\phi)$ may be regarded as an equivalent sphere radius. Some typical values of $\tau(\phi)$ are given in Table 3.

If Equation (34) is substituted into Equation (28), we obtain

$$D_i' = 6\,\pi\,\mu\,\eta\,\tau(\phi)\,U'\left[\,1 + \sigma\frac{\eta\tau(\phi)}{R_o}\right] \qquad (36)$$

Taking note of the equation

$$v = 4/3\,\pi\,\phi\,\eta^3$$

for the volume of an ellipsoid, it follows that

$$\sigma\frac{\eta}{R_o} = \frac{\gamma}{\phi^{1/3}}\,C^{1/3}$$

and

$$D_i' = 6\,\pi\,\mu\,\eta\,\tau(\phi)\,U'\left[\,1 + \left(\gamma\frac{\tau(\phi)}{\phi^{1/3}}\right)C^{1/3}\right] \qquad (37)$$

If U_o' and U_o are, respectively, settling velocities of an ellipsoid and sphere in an infinite medium it can readily be shown that

$$\frac{U'}{U_o'} = \frac{1}{1 + \left(\gamma\frac{\tau(\phi)}{\phi^{1/3}}\right)C^{1/3}} \qquad (38)$$

and

$$\frac{U'}{U_o} = \frac{\phi^{1/3}/\tau(\phi)}{1 + \left(\gamma\frac{\tau(\phi)}{\phi^{1/3}}\right)C^{1/3}} \qquad (39)$$

To illustrate the above results consider a rhombohedral suspension of ellipsoids settling perpendicular to their symmetry axes with $\phi = 100$ and $\tau(\phi) = 23$.

Then

$$\frac{U'}{U_o'} = \frac{1}{1 + 8.87\,C^{1/3}}$$

and

$$\frac{U'}{U_o} = \frac{0.202}{1 + 8.87\,C^{1/3}}$$

The influence of orientation on the drag and rotational moment experienced by interacting ellipsoids has been studied by Wakiya (*20*). He has shown that the lowest order terms representing orientation are to the fourth power of the ratio of semimajor axis to center-to-center distance. Thus orientation need not be considered in a point force approximation, and Equations (38) and (39) are valid representations of settling velocity.

DISCUSSION

The settling velocity equations for ordered suspensions derived in this paper are quite similar to theoretical equations obtained by different methods. The work of Hasimoto (*9*) is of particular interest, as it is an exact point force treatment not requiring successive approximations. Hasimoto considered an infinite periodic array of point forces and solved the creeping motion equations by assuming a Fourier series for the velocity and pressure gradient fields. He obtained a convergent solution in the

Table 3. Values of $\tau(\phi)$ for an Ellipsoid of Revolution from Equation (35)

ϕ	$\tau(\phi)$
1.01	1.004
10.0	3.812
100.0	23.0

absence of container walls by balancing the force acting on a particle by the mean pressure gradient of the fluid. Hasimoto's equation for settling velocity vs. particle concentration in a cubic suspension is

$$\frac{U}{U_o} = \frac{1}{1 + 1.7601\, C^{1/3}} \quad (40)$$

It is our belief that the difference between Hasimoto's coefficient of 1.7601 and 1.92 obtained in the present treatment stems from equating the mean pressure gradient to the force acting on a particle. Brenner and Happel (*1*) have shown that the pressure drop produced by a single particle at the axis of a cylinder is twice the force acting on the particle. Furthermore, the work of Kawaguti (*10*) on the motion of a particle in a hypothetical frictionless cylinder also results in smaller particle resistance. It is therefore conceivable that if the present work had been performed with a frictionless cylinder, the coefficient of 1.76 would have been obtained.

In addition to the work of Hasimoto many cell treatments have been explored in which boundary conditions are satisfied on the particle surface and in the hypothetical envelope surrounding the particle.

Uchida (*19*) and Happel (*7*) employed cell models to obtain the following equations for settling rate at low particle concentrations

Uchida:

$$\frac{U}{U_o} = \frac{1}{1 + 2.1\, C^{1/3}} \quad (41)$$

Happel:

$$\frac{U}{U_o} = \frac{1}{1 + 1.5\, C^{1/3}} \quad (42)$$

Other reflection treatments of sedimentation (*3, 14, 15*) using the cubic model have resulted in equations which are similar to Equation (19). In the work of Burger (*3*) and McNown and Lin (*14*) a convergent solution was obtained by superimposing a velocity field resulting from an assumed continuous distribution of force. The force density was equated to the average pressure gradient. The settling rate equation derived is

$$\frac{U}{U_o} = \frac{1}{1 + 1.6\, C^{1/3}} \quad (43)$$

The criticism of Hasimoto's work with regard to omission of wall force would apply here also.

Perhaps the best known treatment of random suspensions is that of Burgers (*3*). He positioned particles according to a probability distribution function which allowed particles to occupy all positions greater than one diameter from an object particle with equal probability and less than one diameter with zero probability. His equation

$$\frac{U}{U_o} = \frac{1}{1 + 6.88\, C} \quad (44)$$

in which settling rate is a function of the first power of concentration is a direct consequence of this assumed distribution. The principal objection to the distribution function employed by Burgers is that it assumes equal average particle density in every spherical shell outside the object particle. This gives rise to a greater particle density in a small volume containing the object particle, and results in higher settling velocities due to the large effect of particles in its close proximity. The inconsistency of such a distribution is avoided in the present treatment by positioning particles with equal probability at all locations. The correction to Stokes' law is in terms of the one-third power of concentration, because the same technique is employed to sum particle interaction and wall effect as in the ordered suspensions.

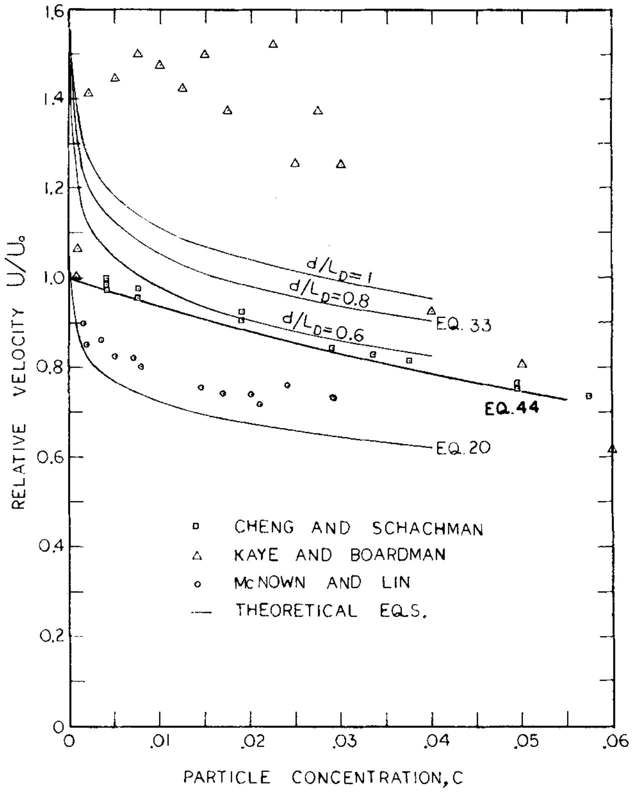

Fig. 6. Comparison of theoretical equation with data in dilute concentration range.

The most consistent aspect of the relationship between the theoretical equations for sedimentation and experimental data in the dilute concentration range, $C < 0.10$, is that experimental settling rates are greater for any given concentration. Figure 6 contains a sampling of experimental data which exhibits large scatter and illustrates the sensitivity to particle characteristics and experimental technique. The highest settling rates are those observed by Kaye and Boardman (*11*). They explained settling rates greater than Stokes' law for their data on the basis of agglomeration, or cluster formation.

It should be stressed that the work reported in this paper is computational and it relies on the postulation of a model for the sedimenting system. The models discussed so far have assumed a uniform distribution of particles: in two cases ordered, as in the cubic and rhombohedral models, and in one case random. A simple model of a suspension containing clusters of particles may be constructed by placing doublets in place of single spheres. The utility of such a model is that it should reveal the direction and approximate magnitude of changes in settling rate due to cluster formation. In an actual suspension where large clusters are possible, we would expect the effect to be in the same direction, although more pronounced.

The model selected for the study of cluster formation consisted of vertically oriented doublets in place of single spheres in the rhombohedral arrangement. Values of λ taken from reference *8* are substituted into Equation (33), γ is set equal to 1.79, and a family of settling rate curves is produced for different values of the parameter d/L_D, as in Figure 6. It should be noted that these curves would have been displaced downward for a suspension of horizontally oriented doublets. However, they still would predict higher settling rates than the curve for a rhombohedral suspension of spherical particles, also shown in Figure 6 (labeled *EQ 20*).

It is apparent from Figure 6 that if one assumes various degrees of cluster formation in a rhombohedral model, ranging from uniformly distributed single spheres to uniformly distributed touching vertical doublets, widely divergent settling rates are obtained. It is our belief that a model containing larger size clusters could be constructed to yield settling rates as high as those observed by Kaye and Boardman (*11*).

It should be noted that the curves in Figure 6 are for constant values of the parameter d/L_D. This is equivalent to a constant cluster size over the concentration range covered. In an actual suspension the authors believe that the tendency toward cluster formation will depend on particle concentration, in which case the coefficient of $C^{1/3}$ will itself be a function of C. The nature of this function can only be obtained by working with a model which does not rely on an assumed distribution of particles, but is capable of predicting particle distribution from the properties of the suspension.

SUMMARY

A superposition technique is employed to obtain a first-order correction to Stokes' settling velocity for dilute suspensions. It is shown that agglomeration can markedly increase settling velocity in the dilute range and hence must be included in theoretical or experimental settling rate equations.

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NOTATION

a = sphere radius
C = particle concentration
d = sphere diameter
D = drag acting on a sphere in a suspension
D' = drag acting on an arbitrary particle in a suspension
D_o = drag acting on a sphere in an infinite medium
D_o' = drag acting on an arbitrary particle in an infinite medium
H = height of tetrahedra in rhombohedral suspension
L = side of tetrahedra in rhombohedral suspension
L_d = center-to-center distance between spheres in doublet
L_{ji} = distance from particle j to position i
p = fluid pressure
R_o = cylinder radius
R_i = horizontal distance from cylinder axis to position i
U = settling velocity of a sphere in a suspension
U' = settling velocity of an arbitrary particle in a suspension
U_o = settling velocity of a sphere in an infinite medium
U_o' = settling velocity of an arbitrary particle in an infinite medium
$\mathbf{v}$ = fluid velocity
W' = fluid velocity reflected from the container wall due to the presence of an arbitrary particle
W_{ji} = fluid motion equal to $-\omega_{ji}$ at cylinder

Greek Letters

α = suspension density, number of particles per unit volume
β = radial eccentricity, R_i/R_o
γ = concentration coefficient in settling rate equation
ζ = ellipsoid semimajor axis
η = ellipsoid semiminor axis
λ = resistance coefficient for doublets
μ = fluid viscosity
π = 3.1416
ρ = fluid density
σ = particle size coefficient in settling rate equation
ϕ = ratio ζ/η
ω_{ji} = fluid motion induced by particle j, evaluated at position i. Equal to U at particle j

Mathematical Operators

∇ = gradient =
$$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

∇^2 = Laplacian =
$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\nabla \cdot \mathbf{v}$ =
$$\text{div } \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

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